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The slow motion of a rigid particle in a second-order fluid

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The purpose of the present paper is to reach some general conclusions on the motion of rigid particles in a homogeneous shear flow of a viscoelastic fluid. Under the basic assumption of nearly Newtonian slow flow, the creeping-motion equations for a second-order fluid with characteristic time constants $\kappa_0^{(2)}$ and $\kappa_0^{(11)}$ can be employed. It is shown that the $\kappa_0^{(2)}$ contributions to the hydrodynamic force $\mathbf{F}$ and couple $\mathbf{G}$ depend upon the hydrodynamic force, couple and stresslet which act upon the particle in a Newtonian fluid (termed $\mathbf{F}^{(1)}$, $\mathbf{G}^{(1)}$ and $\mathbf{S}^{(1)}$, respectively). Since this relation involves time derivatives of $\mathbf{F}^{(1)}$ and $\mathbf{G}^{(1)}$, a little reflexion is needed to realize that the modification of the classical Stokes law for steady translation in a quiescent fluid can have no $\kappa_0^{(2)}$ term. Since no results of such generality are possible for the $\kappa_0^{(11)}$ contributions we focus attention on transversely isotropic particles. Employing the concept of material tensors, the symmetry of such particles dictates the form these tensors adopt. This alone is sufficient to show that sedimentation in a quiescent fluid is accompanied by a change in orientation until a stable terminal orientation is attained. Depending upon the type of particle only one of the two orientations, axis of symmetry parallel or perpendicular to the external force, is stable. Another result concerns two-dimensional shear flow, for which we show that the symmetry axis has to drift through various Jeffery orbits until an equilibrium orientation is reached. While the orbits $C = 0$ and $C = \infty$ are equilibrium orbits for every transversely isotropic particle there may be a third such preferred orbit, which we denote by C^*. In order for these orbits to be stable certain restrictions have to hold, showing that the orbits $C = 0$ and C^* cannot both be stable. For the special case of a rigid tridumbbell of axis ratio s the orbit C^* does not exist. If $s > 1$ the drift for this particle is into the orbit $C = 0$ while for $s < 1$ it is into the orbit $C = \infty$. This agrees qualitatively quite well with experimental results obtained for rods and disks. No quantitative comparison is possible; the particle shape influences the result quantitatively owing to its effect on the combination of the fluid parameters $\kappa_0^{(2)}$ and $\kappa_0^{(11)}$.

1. Introduction

The behaviour of isolated rigid particles submerged in a Newtonian fluid has received much attention in the literature. If the motion is sufficiently slow the hydrodynamic force as well as the couple relative to some point O inside the body can depend only linearly on the parameters of the problem. As long as the motion of the fluid can be approximated by a homogeneous shear flow, these parameters are the relative translational velocity $\tilde{\mathbf{U}}_0$, the relative angular velocity $\tilde{\mathbf{\Omega}}$ and a pure strain $\mathbf{E}$. The tensors relating these parameters to the force and the couple are material tensors since apart from their dependence on the arbitrary choice of the point O they depend only upon the exterior geometry of the particle surface. Any symmetries the body possesses reduces the number of independent coefficients in these tensors. In this way some generally valid conclusions can be obtained. Examples of such conclusions include the settling of homogeneous transversely isotropic particles without variation of the initial orientation or the behaviour of such particles in simple shear. With the exception of certain very long bodies the rotation in such a flow is identical to that of some ellipsoid of revolution (Bretherton 1962). That is to say, the symmetry axis of a transversely isotropic particle rotates periodically around the vorticity axis without changing its orbit.

A recent calculation by Leal (1975) shows that slender rod-like particles immersed in a second-order fluid behave differently. Not only do they attain a preferred orientation by settling in a gravitational field but they also change their orbit in a two-dimensional shear flow until the particle axis is parallel to the vorticity axis (orbit constant $C = 0$). Experiments at low shear rates for rods agree with the latter result (Gauthier, Goldsmith & Mason 1971). These experiments also show that small disks change their orbit too. This time the axis of revolution drifts into the plane of the shear (orbit constant $C = \infty$). Consequently, the question arises as to whether orbit drift (and possibly also the preferred terminal orientation in an external force field) is characteristic of all transversely isotropic particles suspended in a viscoelastic fluid.

In order to obtain an answer we assume the flow to be rheologically so slow that the stress tensor is related to the rate of strain via the constitutive equation of a second-order fluid. It may therefore be characterized by two time constants: $\kappa_0^{(2)}$ and $\kappa_0^{(11}$ (Giesekus 1963). Furthermore we confine our attention to homogeneous flows, i.e. flows for which the (undisturbed) velocity gradient is identically constant. Denoting by $\mathbf{F}^{(1)}$, $\mathbf{G}^{(1)}$ and $\mathbf{S}^{(1)}$ the hydrodynamic force, couple and stresslet which are exerted upon the particle in a Newtonian fluid, the $\kappa_0^{(2)}$ contributions to the force $\mathbf{F}$ and couple $\mathbf{G}$ are shown in §2 to depend only upon $\mathbf{F}^{(1)}$, $\mathbf{G}^{(1)}$, $\mathbf{S}^{(1)}$ and $\partial\mathbf{F}^{(1)}/\partial t$ or $\partial\mathbf{G}^{(1)}/\partial t$, respectively. When particle inertia is neglected these time derivatives vanish for transversely isotropic particles, if the particles are either freely suspended or if the fluid is quiescent and a constant external force $\mathbf{F}^e$ is acting.

For the $\kappa_0^{(11)}$ contributions to $\mathbf{F}$ and $\mathbf{G}$ we rely on the quadratic dependence of these quantities upon the parameters of the problem. Since this is the case the relation can again involve only intrinsic material tensors. And by confining our attention to transversely isotropic particles we can write down the forms these tensors adopt (§3).

On adding up all contributions we see that the centre of symmetry of a freely suspended transversely isotropic particle will always move with the local fluid velocity while its orientation will change depending upon the type of flow. For the important case of a two-dimensional shear flow the axis of symmetry will rotate periodically around the vorticity axis. In doing so it continuously changes its orbit until it reaches an equilibrium orbit. Although the orbits $C = 0$ and $C = \infty$ are equilibrium orbits for any transversely isotropic particle there may be a third such preferred orbit, called C^*. Certain restrictions have to be met for these orbits to be stable. These arguments show that the orbits $C = 0$ and C^* cannot both be stable, so that for any given particle there can be at most two stable equilibrium orientations. All this is demonstrated in §4, where we also reach the conclusion that a transversely isotropic particle subjected only to a force will in general rotate. In a quiescent fluid the initial orientation will change until the particle attains its spin-free terminal state. In this state the axis of symmetry can be either parallel or perpendicular to the direction of the force, depending upon the type of particle.

In order to be more specific about these general results a rigid tridumbbell with two axes equal is studied in detail (§5). Employing the general transformation laws (appendix A), we use the results obtained previously for a sphere (Brunn 1976*b*) to calculate all the material tensors. Since we neglect any hydrodynamic interaction between the ends of the dumbbells the translational behaviour of a rigid tridumbbell is the same as that of a spherically isotropic body. Thus in sedimentation no change in the initial orientation can occur. On the other hand the rotational behaviour is that of a transversely isotropic particle. In a simple shear flow only the equilibrium orbits $C = 0$ and $C = \infty$ exist and the tridumbbell will drift into the orbit $C = 0$ if its axis ratio s is larger than one and into the orbit $C = \infty$ for $s < 1$. Qualitatively this agrees quite well with the experiments cited above. Since these experiments were performed on rods and disks no quantitative agreement is possible: the shape of the particle determines the magnitude of the effect owing to its influence on the combination of the time constants $\kappa_0^{(2)}$ and $\kappa_0^{(11)}$.

2. Formulation of the problem

Consider an incompressible fluid of viscosity η whose state of motion $(\mathbf{U}, P)$ satisfies the steady-state creeping-motion equations

$$\frac{\partial}{\partial \mathbf{r}}.\mathbf{U} = 0, \quad -\frac{\partial}{\partial \mathbf{r}}P + \eta\nabla^2\mathbf{U} = 0. \tag{2.1}$$

Suppose now that a rigid particle of arbitrary shape (surface S_0, volume V_0) is placed in that fluid. If the fluid is Newtonian and if we denote by $\mathbf{u}_0^{(1)}$ the translational velocity of some point O of the particle and by $\boldsymbol{\omega}^{(1)}$ the angular velocity of the particle with respect to a fixed laboratory frame, the inertia-less $(\mathbf{u}^{(1)}, p^{(1)})$ fields are known to be

$$\mathbf{u}^{(1)}(\mathbf{r}) = \mathbf{U}(\mathbf{r}) - \frac{1}{8\pi\eta}\int_{S_0} d^2x_s\, \mathbf{n}\,.\boldsymbol{\tau}^{(1)}.\left\{\frac{\boldsymbol{\delta}}{|\mathbf{r}-\mathbf{r}_s|} + \frac{(\mathbf{r}-\mathbf{r}_s)(\mathbf{r}-\mathbf{r}_s)}{|\mathbf{r}-\mathbf{r}_s|^3}\right\}, \tag{2.2a}$$

$$p^{(1)}(\mathbf{r}) = P - \frac{1}{4\pi}\int_{S_0} d^2x_s\, \mathbf{n}\,.\boldsymbol{\tau}^{(1)}.\frac{\mathbf{r}-\mathbf{r}_s}{|\mathbf{r}-\mathbf{r}_s|}, \tag{2.2b}$$

where $\boldsymbol{\tau}^{(1)}$ denotes the stress tensor $\boldsymbol{\tau}^{(j)}$ with $j = 1$, which is of the form

$$\boldsymbol{\tau}^{(j)} = -\frac{\partial p^{(j)}}{\partial \mathbf{r}} + \eta\left[\frac{\partial \mathbf{u}^{(j)}}{\partial \mathbf{r}} + \left(\frac{\partial \mathbf{u}^{(j)}}{\partial \mathbf{r}}\right)^{\dagger}\right]. \tag{2.3}$$

If $\mathbf{r}_0$ is the instantaneous position vector of O, measured like $\mathbf{r}$ from one point fixed in space, the $(\mathbf{u}^{(1)}, p^{(1)})$ fields at large distances from the particle become

$$\begin{aligned}\mathbf{u}^{(1)}(\mathbf{r}) = \mathbf{U}(\mathbf{r}) &- \frac{1}{8\pi\eta\,|\mathbf{r}-\mathbf{r}_0|}\mathbf{F}^{(1)}.\left[\boldsymbol{\delta} + \frac{(\mathbf{r}-\mathbf{r}_0)(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^2}\right]\\ &- \frac{1}{8\pi\eta\,|\mathbf{r}-\mathbf{r}_0|^2}\left\{\mathbf{G}_0^{(1)}\times\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|} + 3\mathbf{S}_0^{(1)}:\frac{(\mathbf{r}-\mathbf{r}_0)(\mathbf{r}-\mathbf{r}_0)(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^3}\right\} + O\left(\frac{1}{|\mathbf{r}-\mathbf{r}_0|^3}\right),\end{aligned} \tag{2.4a}$$

$$p^{(1)}(\mathbf{r}) = P - \frac{1}{4\pi|\mathbf{r}-\mathbf{r}_0|^2}\mathbf{F}^{(1)}.\frac{(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^2} - \frac{3}{4\pi|\mathbf{r}-\mathbf{r}_0|^3}\mathbf{S}_0^{(1)}:\frac{(\mathbf{r}-\mathbf{r}_0)(\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^2} + O\left(\frac{1}{|\mathbf{r}-\mathbf{r}_0|^4}\right). \tag{2.4b}$$

In these equations

$$\mathbf{F}^{(1)} = \int_{S_0} d^2x\, \mathbf{n}\,.\,\boldsymbol{\tau}^{(1)}, \tag{2.5a}$$

$$\mathbf{G}_0^{(1)} = \int_{S_0} d^2x(\mathbf{r}-\mathbf{r}_0)\times\mathbf{n}\,.\,\boldsymbol{\tau}^{(1)}, \tag{2.5b}$$

$$\mathbf{S}_0^{(1)} = \int_{S_0} d^2x\,[\tfrac{1}{2}(\mathbf{r}-\mathbf{r}_0)\,\mathbf{n}\,.\,\boldsymbol{\tau}^{(1)} + \tfrac{1}{2}\mathbf{n}\,.\,\boldsymbol{\tau}^{(1)}(\mathbf{r}-\mathbf{r}_0) - \tfrac{1}{3}\boldsymbol{\delta}\mathbf{n}\,.\,\boldsymbol{\tau}^{(1)}\,.\,(\mathbf{r}-\mathbf{r}_0)] \tag{2.5c}$$

are the hydrodynamic force, couple and stresslet with respect to O which are exerted upon the particle. As far as $\mathbf{F}^{(1)}$ and $\mathbf{G}_0^{(1)}$ are concerned we may replace the integration over the particle surface by an integration over any arbitrary surface S completely surrounding the particle.

For the special case in which $\mathbf{U}$ is a homogeneous velocity field in the sense that the rate-of-deformation tensor is identically constant, $\mathbf{F}^{(1)}$, $\mathbf{G}_0^{(1)}$ and $\mathbf{S}_0^{(1)}$ are known to be

$$\mathbf{F}^{(1)} = \eta\{{}^t\mathbf{K}\,.\,(\mathbf{U}_0-\mathbf{u}_0^{(1)}) + {}^t\mathbf{R}\,.\,(\boldsymbol{\Omega}-\boldsymbol{\omega}^{(1)}) + {}^t\mathbf{Q}\colon\mathbf{E}\}, \tag{2.6a}$$

$$\mathbf{G}_0^{(1)} = \eta\{(\mathbf{U}_0-\mathbf{u}_0^{(1)})\,.\,{}^t\mathbf{R} + {}^r\mathbf{R}\,.\,(\boldsymbol{\Omega}-\boldsymbol{\omega}^{(1)}) + {}^r\mathbf{Q}\colon\mathbf{E}\}, \tag{2.6b}$$

$$\mathbf{S}_0^{(1)} = \eta\{(\mathbf{U}_0-\mathbf{u}_0^{(1)})\,.\,{}^t\mathbf{Q} + (\boldsymbol{\Omega}-\boldsymbol{\omega}^{(1)})\,.\,{}^r\mathbf{Q} + \mathbf{D}\colon\mathbf{E}\}, \tag{2.6c}$$

with

$$\boldsymbol{\Omega} = \frac{1}{2}\frac{\partial}{\partial\mathbf{r}}\times\mathbf{U}, \quad \mathbf{E} = \frac{1}{2}\left[\frac{\partial}{\partial\mathbf{r}}\mathbf{U} + \left(\frac{\partial}{\partial\mathbf{r}}\mathbf{U}\right)^{\dagger}\right], \tag{2.7}$$

the angular velocity and the pure strain of the undisturbed flow as seen by an observer fixed in space. Since in such a flow the particle will in general translate and rotate in a time-dependent manner, the approach velocity $\mathbf{U}_0 = \mathbf{U}(\mathbf{r}_0)$ will in general depend upon time ($\mathbf{r}_0 = \mathbf{r}_0(t)$). For convenience, let us introduce the abbreviations

$$\tilde{\mathbf{U}}_0 = \mathbf{U}_0 - \mathbf{u}_0^{(1)}, \quad \tilde{\boldsymbol{\Omega}} = \boldsymbol{\Omega} - \boldsymbol{\omega}^{(1)}. \tag{2.8}$$

The tensors appearing in (2.6) are material tensors. They uniquely characterize the particle since they depend only upon its size and shape. Except for ${}^t\mathbf{K}$ all tensors depend upon the chosen reference point O. Apart from the trivial symmetries these tensors have (e.g. D_{ijkl} is symmetric and irreducible in its first two and in its last two indices) there are also certain 'kinetic' symmetry relations, some of which have already been incorporated into (2.6) (note that only six independent tensors appear). The other relations are (Hinch 1972)

$${}^tK_{ij} = {}^tK_{ji}, \quad {}^rR_{ij} = {}^rR_{ji}, \quad D_{ijkl} = D_{klij}. \tag{2.9}$$

So far, we have assumed the fluid to be Newtonian. If we drop that restriction and consider a viscoelastic fluid the $(\mathbf{u}, p)$ fields as well as the state of motion of the particle $(\mathbf{u}_0, \boldsymbol{\omega})$ have to be modified even if we use the same undisturbed fields $(\mathbf{U}, P)$. It we split $(\mathbf{u}, p)$ into $(\mathbf{u}^{(1)}+\mathbf{u}^{2)}, p^{(1)}+p^{(2)})$ and $(\mathbf{u}_0, \boldsymbol{\omega})$ into $(\mathbf{u}_0^{(1)}+\mathbf{u}_0^{(2)}, \boldsymbol{\omega}^{(1)}+\boldsymbol{\omega}^{(2)})$ and consider all fields with a superscript 2 as perturbations, a decomposition of the stress tensor of the form

$$\mathbf{T} = \mathbf{T}^{(1)} + \mathbf{T}^{(2)} = \boldsymbol{\tau}^{(1)} + (\boldsymbol{\tau}^{(2)} + \boldsymbol{\sigma}^{(1)}) \tag{2.10}$$

with $\boldsymbol{\sigma}^{(1)}$ determined solely by the $(\mathbf{u}^{(1)}, p^{(1)})$ fields seems appropriate. This decomposition will lead to an obvious modification of the force and torque and the terms which have to be added to $\mathbf{F}^{(1)}$ and $\mathbf{G}_0^{(2)}$ will be denoted by $\mathbf{F}^{(2)}$ and $\mathbf{G}_0^{(2)}$, respectively.

In particular, if we employ the model of a second-order fluid we have (Giesekus 1963)

$$\boldsymbol{\sigma}^{(1)} = 2\eta[\kappa_0^{(11)}\mathbf{f}^{(1)}.\mathbf{f}^{(1)} + \kappa_0^{(2)}\mathbf{f}^{(2)}], \tag{2.11}$$

with

$$\mathbf{f}^{(1)} = \frac{1}{2}\left[\frac{\partial \mathbf{u}^{(1)}}{\partial \mathbf{r}} + \left(\frac{\partial \mathbf{u}^{(1)}}{\partial \mathbf{r}}\right)^{\dagger}\right], \quad \boldsymbol{\omega}^{(1)} = \frac{1}{2}\left[\frac{\partial \mathbf{u}^{(1)}}{\partial \mathbf{r}} - \left(\frac{\partial \mathbf{u}^{(1)}}{\partial \mathbf{r}}\right)^{\dagger}\right], \tag{2.12a}$$

$$\mathbf{f}^{(2)} = \frac{\partial}{\partial t}\mathbf{f}^{(1)} + \mathbf{u}^{(1)}.\frac{\partial}{\partial \mathbf{r}}\mathbf{f}^{(1)} + \boldsymbol{\omega}^{(1)}.\mathbf{f}^{(1)} - \mathbf{f}^{(1)}.\boldsymbol{\omega}^{(1)}. \tag{2.12b}$$

In what follows it will prove convenient to consider the $\kappa_0^{(11)}$ and $\kappa_0^{(2)}$ contributions separately, which requires an additional decomposition of the $(\mathbf{u}^{(2)}, p^{(2)})$ fields into $(\mathbf{u}^{(21)} + \mathbf{u}^{(22)}, p^{(21)} + p^{(22)})$ and a similar decomposition of $\mathbf{u}_0^{(2)}$ and $\boldsymbol{\omega}^{(2)}$.

2.1. *The $\kappa_0^{(11)}$ contributions*

By definition we have

$$\mathbf{F}^{(21)} = \int_S d^2x\, \mathbf{n}.(\boldsymbol{\tau}^{(21)} + 2\eta\kappa_0^{(11)}\mathbf{f}^{(1)}.\mathbf{f}^{(1)}),$$

$$\mathbf{G}_0^{(21)} = \int_S d_2x(\mathbf{r}-\mathbf{r}_0)\times\mathbf{n}.(\boldsymbol{\tau}^{(21)} + 2\eta\kappa_0^{(11)}\mathbf{f}^{(1)}.\mathbf{f}^{(1)}), \tag{2.13}$$

where S is any arbitrary surface completely surrounding the particle. But from (2.2) we know

$$\mathbf{f}^{(1)} = \mathbf{E} + \tilde{\mathbf{f}}^{(1)},$$

where $\tilde{\mathbf{f}}^{(1)}$ depends only upon the quantities $\tilde{\mathbf{U}}_0$, $\tilde{\boldsymbol{\Omega}}$ and $\mathbf{E}$. By virtue of the quadratic dependence of $(\mathbf{u}^{(21)}, p^{(21)})$ upon these quantities $\mathbf{F}^{(21)}$ and $\mathbf{G}_0^{(21)}$ must therefore be of the form

$$\begin{aligned}\mathbf{F}^{(21)} = \eta\{&-{}^t\mathbf{K}.\mathbf{u}_0^{(21)} - {}^t\mathbf{R}.\boldsymbol{\omega}^{(21)} + {}^t\mathbf{K}^{(2)}:\tilde{\mathbf{U}}_0\tilde{\mathbf{U}}_0\\ &+ {}^t\mathbf{C}^{(21)}:\tilde{\mathbf{U}}_0\tilde{\boldsymbol{\Omega}} + {}^t\mathbf{C}^{(22)}\vdots\,\tilde{\mathbf{U}}_0\mathbf{E} + {}^t\mathbf{R}^{(2)}:\tilde{\boldsymbol{\Omega}}\tilde{\boldsymbol{\Omega}} + {}^t\mathbf{C}^{(31)}\vdots\,\tilde{\boldsymbol{\Omega}}\mathbf{E} + {}^t\mathbf{Q}^{(2)}\vdots\,\mathbf{E}\mathbf{E}\},\end{aligned} \tag{2.14}$$

$$\begin{aligned}\mathbf{G}_0^{(21)} = \eta\{&-\mathbf{u}_0^{(21)}.{}^t\mathbf{R} - {}^r\mathbf{R}.\boldsymbol{\omega}^{(21)} + {}^r\mathbf{K}^{(2)}:\tilde{\mathbf{U}}_0\tilde{\mathbf{U}}_0\\ &+ {}^r\mathbf{C}^{(21)}:\tilde{\mathbf{U}}_0\tilde{\boldsymbol{\Omega}} + {}^r\mathbf{C}^{(22)}\vdots\,\tilde{\mathbf{U}}_0\mathbf{E} + {}^r\mathbf{R}^{(2)}:\tilde{\boldsymbol{\Omega}}\tilde{\boldsymbol{\Omega}} + {}^r\mathbf{C}^{(23)}\vdots\,\tilde{\boldsymbol{\Omega}}\mathbf{E} + {}^r\mathbf{Q}^{(2)}\vdots\,\mathbf{E}\mathbf{E}\}.\end{aligned} \tag{2.15}$$

The tensors appearing in these equations are again material tensors and as such have the same properties as the tensors appearing in (2.6). Apart from ${}^t\mathbf{K}$ and ${}^t\mathbf{K}^{(2)}$ they all depend upon the choice of the reference point O.

2.2. *The $\kappa_0^{(2)}$ contributions*

By definition we have

$$\mathbf{F}^{(22)} = \int_S d^2x\, \mathbf{n}.(\boldsymbol{\tau}^{(22)} + 2\eta\kappa_0^{(2)}\mathbf{f}^{(2)}), \tag{2.16a}$$

$$\mathbf{G}_0^{(22)} = \int_S d^2x(\mathbf{r}-\mathbf{r}_0)\times\mathbf{n}.(\boldsymbol{\tau}^{(22)} + 2\eta\kappa_0^{(2)}\mathbf{f}^{(2)}), \tag{2.16b}$$

where $\mathbf{f}^{(2)}$ is the corotational derivative of $\mathbf{f}^{(1)}$ [cf. (2.12)]. As such

$$\frac{\partial}{\partial \mathbf{r}}.\mathbf{f}^{(2)}$$

is an irrotational vector field and can be written as the gradient of a scalar field involving in a known manner (Giesekus 1963) only the $(\mathbf{u}^{(1)}, p^{(1)})$ field. With $\boldsymbol{\tau}^{(22)}$ thus determined (2.16a, b) become

$$\mathbf{F}^{(22)} = \eta\{-{}^t\mathbf{K}.\mathbf{u}_0^{(22)} - {}^t\mathbf{R}.\boldsymbol{\omega}^{(22)}\} + \eta\kappa_0^{(2)}\int_S d^2x\left\{2\mathbf{n}.\mathbf{f}^{(2)} - \frac{1}{\eta}\mathbf{n}\frac{\partial}{\partial t}p^{(1)} - \mathbf{n}\mathbf{u}^{(1)}.\nabla^2\mathbf{u}^{(1)} - \mathbf{n}\mathbf{f}^{(1)}:\mathbf{f}^{(1)}\right\}, \quad (2.17a)$$

$$G_0^{(22)} = \eta\{-\mathbf{u}_0^{(22)}.{}^t\mathbf{R} - {}^r\mathbf{R}.\boldsymbol{\omega}^{(22)}\} + \eta\kappa_0^{(2)}\int_S d^2x(\mathbf{r}-\mathbf{r}_0) \times\left\{2\mathbf{n}.\mathbf{f}^{(2)} - \frac{1}{\eta}\mathbf{n}\frac{\partial}{\partial t}p^{(1)} - \mathbf{n}\mathbf{u}^{(1)}.\nabla^2\mathbf{u}^{(1)} - \mathbf{n}\mathbf{f}^{(1)}:\mathbf{f}^{(1)}\right\}. \quad (2.17b)$$

In particular, if S denotes a space-fixed surface which momentarily coincides with a sphere of infinite radius around O the expansion (2.4) is sufficient for the result

$$\mathbf{F}^{(22)} = \eta\{-{}^t\mathbf{K}.\mathbf{u}_0^{(22)} - {}^t\mathbf{R}.\boldsymbol{\omega}^{(22)}\} + \kappa_0^{(2)}\left\{\mathbf{F}^{(1)}\times\boldsymbol{\Omega} + \mathbf{E}.\mathbf{F}^{(1)} + \int_S d^2x\,\mathbf{n}.\frac{\partial}{\partial t}\boldsymbol{\tau}^{(1)}\right\}, \quad (2.18a)$$

$$\mathbf{G}_0^{(22)} = \eta\{-\mathbf{u}_0^{(22)}.{}^t\mathbf{R} - {}^r\mathbf{R}.\boldsymbol{\omega}^{(22)}\} + \kappa_0^{(2)}\left\{\mathbf{F}^{(1)}\times\mathbf{U}_0 + \mathbf{G}_0^{(1)}\times\boldsymbol{\Omega} + 2\boldsymbol{\epsilon}:(\mathbf{E}.\mathbf{S}_0^{(1)}) + \int_S d^2x(\mathbf{r}-\mathbf{r}_0)\times\mathbf{n}.\frac{\partial}{\partial t}\boldsymbol{\tau}^{(1)}\right\} \quad (2.18b)$$

to emerge. Recalling the remarks following (2.5) we can rewrite (2.18) as

$$\mathbf{F}^{(22)} = \eta\{-{}^t\mathbf{K}.\mathbf{u}_0^{(22)} - {}^t\mathbf{R}.\boldsymbol{\omega}^{(22)}\} + \kappa_0^{(2)}\{\mathbf{F}^{(1)}\times\boldsymbol{\Omega} + \partial\mathbf{F}^{(1)}/\partial t + \mathbf{E}.\mathbf{F}^{(1)}\}, \quad (2.19)$$

$$\mathbf{G}_0^{(22)} = \eta\{-\mathbf{u}_0^{(22)}.{}^t\mathbf{R} - {}^r\mathbf{R}.\boldsymbol{\omega}^{(22)}\} + \kappa_0^{(2)}\{\mathbf{F}^{(1)}\times\tilde{\mathbf{U}}_0 + \mathbf{G}_0^{(1)}\times\boldsymbol{\Omega} + \partial\mathbf{G}_0^{(1)}/\partial t + 2\boldsymbol{\epsilon}:(\mathbf{E}.\mathbf{S}_0^{(1)})\}, \quad (2.20)$$

where use has been made of the relation $\partial\mathbf{r}_0/\partial t = \mathbf{u}_0$. Without going into too much detail, a few remarks are in order.

First of all, if the particle is held fixed the term $\partial\mathbf{F}^{(1)}/\partial t$, the time rate of change of $\mathbf{F}^{(1)}$ as seen by an observer fixed in space, vanishes and the same is true for $\partial\mathbf{G}_0^{(1)}/\partial t$. In particular, if the undisturbed motion is merely a steady translation and rotation ($\mathbf{E} = 0$) we get

$$\mathbf{F}^{(22)}.\mathbf{U}_0 + \mathbf{G}_0^{(22)}.\boldsymbol{\Omega} = 0, \quad (2.21)$$

a relation which Caswell (1968) derived from energy considerations.

On the other hand, if under the action of a constant external force the particle is known to translate steadily without rotation in a quiescent Newtonian fluid (cf. §4.1) then

$$\mathbf{F}^{(22)} = -\eta\{{}^t\mathbf{K}.\mathbf{u}_0^{(22)} + {}^t\mathbf{R}.\boldsymbol{\omega}^{(22)}\}, \quad (2.22a)$$

$$\mathbf{G}_0^{(22)} = -\eta\{\mathbf{u}_0^{(22)}.{}^t\mathbf{R} + {}^r\mathbf{R}.\boldsymbol{\omega}^{(22)}\} - \kappa_0^{(2)}\mathbf{F}^{(1)}\times\mathbf{u}_0^{(1)}. \quad (2.22b)$$

Consequently, in this situation the concept of material tensors still applies to the $\kappa_0^{(2)}$ contributions. It is interesting to see that no terms quadratic in $\mathbf{u}_0^{(1)}$ appear in the expression for $\mathbf{F}^{(22)}$; the modification of the classical Stokes law can therefore have at most a $\kappa_0^{(11)}$ term. This result is at odds with the result in Leal (1975), indicating an error in Leal's calculation.

Second, by arguments entirely analogous to those which led us to (2.14), for a moving particle we should expect time derivatives of $\mathbf{U}_0$ and $\mathbf{\Omega}$ to appear in the expressions for $\mathbf{F}^{(22)}$ and $\mathbf{G}_0^{(22)}$. But (2.19) and (2.20) tell us that these terms appear only indirectly, namely via $\partial\mathbf{F}^{(1)}/\partial t$ and $\partial\mathbf{G}_0^{(1)}/\partial t$ respectively, and this is of great advantage. For, if $\mathbf{F}^{(1)}$ and $\mathbf{G}_0^{(1)}$ are known to vanish identically, we have

$$\mathbf{F}^{(22)} = -\eta\{{}^t\mathbf{K}.\mathbf{u}_0^{(22)} + {}^t\mathbf{R}.\boldsymbol{\omega}^{(22)}\}, \tag{2.23a}$$

$$\mathbf{G}_0^{(22)} = -\eta\{\mathbf{u}_0^{(22)}.{}^t\mathbf{R} + {}^r\mathbf{R}.\boldsymbol{\omega}^{(22)}\} + 2\kappa_0^{(2)}\boldsymbol{\epsilon}:(\mathbf{E}.\mathbf{S}_0^{(1)}). \tag{2.23b}$$

As an example consider a homogeneous transversely isotropic particle which is neutrally buoyant. Since particle inertia has to be neglected for reasons of consistency the conditions $\mathbf{F}^{(1)} \equiv 0$ and $\mathbf{G}_0^{(1)} \equiv 0$ are clearly met,† and it is such particles which we shall examine in more detail.

3. The material tensors for a transversely isotropic particle

Consider a particle with three mutually perpendicular symmetry planes and let O denote the point of intersection of these three planes. Choosing these planes to be the planes Ox_1x_2, Ox_1x_3 and Ox_2x_3, we shall call the particle transversely isotropic if two of the co-ordinate axes, say Ox_2 and Ox_3, are indistinguishable. Examples are rigid tridumbbells with two equal axes (§ 4) or bodies of revolution with fore–aft symmetry. Let $\mathbf{e}$ be a unit vector parallel to the symmetry axis (the Ox_1 axis) and let the components of $\mathbf{e}$ in an arbitrary Cartesian co-ordinate system be e_j, $j = 1, 2, 3$. All material tensors at O must be then expressible as a combination of the tensors δ_{ij} and ϵ_{ijk} and an even number of the e_j. A distinction must be made between tensors and pseudo-tensors since if a material tensor of rank l with the superscript t is indeed a tensor (of parity $(-1)^l$) the corresponding quantity with superscript r will be a pseudo-tensor (of parity $(-1)^{l+1}$) and vice versa. If we denote by $A_{i_1\ldots i_l}$ a transversely isotropic tensor of rank l and by $B_{i_1\ldots i_l}$ a transversely isotropic pseudo-tensor of rank l these quantities must be of the following form:

$$A_{ij} = A_1(\delta_{ij} - e_i e_j) + A_2 e_i e_j, \tag{3.1}$$

$$B_{ij} = 0, \tag{3.2}$$

$$A_{ijk} = 0, \tag{3.3}$$

$$B_{ijk} = B_1\epsilon_{ijk} + B_2\epsilon_{ij\mu}e_\mu e_k + B_3\epsilon_{ik\mu}e_\mu e_j + B_4\epsilon_{jk\mu}e_\mu e_i. \tag{3.4}$$

In particular if $B_{ijk} = B_{ikj}$ then (3.4) reduces to

$$B_{ijk} = 2B_2\Delta^{(2)}_{jk,\mu\nu}\epsilon_{i\mu\eta}e_\eta e_\nu, \tag{3.5}$$

where the irreducible fourth-rank tensor $\mathbf{\Delta}^{(2)}$ is given by (A 6).

The tensor and pseudo-tensor of fourth order which are symmetric and irreducible with respect to the indices j and k are

$$B_{ijkl} = 0, \tag{3.6}$$

$$\begin{aligned} A_{ijkl} = {} & A_1[\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} - 2\delta_{il}e_j e_k] + A_2[\delta_{il}\delta_{jk} - 3\delta_{il}e_k e_j] \\ & + A_3[\delta_{kl}e_i e_j + \delta_{jl}e_i e_k - 2e_i e_j e_k e_l] \\ & + A_4[\delta_{jk}e_i e_l - 3e_i e_j e_k e_l] + A_5[\delta_{ij}e_k e_l + \delta_{ik}e_j e_l - 2e_i e_j e_k e_l]. \end{aligned} \tag{3.7}$$

† Note that external influences ($\mathbf{F}^e$, $\mathbf{G}_0^e$) do not depend upon $\kappa_0^{(11)}$ or $\kappa_0^{(2)}$. Consequently we have to set $\mathbf{F}^{(1)} = -\mathbf{F}^e$, $\mathbf{F}^{(2)} = 0$ and similarly $\mathbf{G}_0^{(1)} = -\mathbf{G}_0^e$, $\mathbf{G}_0^{(2)} = 0$.

Material tensor	Given by	Coefficients
${}^tK_{ij}$	(3.1)	$A_1 = K_\perp,\quad A_2 = K_\parallel$
${}^tR_{ij}$	(3.2)	—
${}^rR_{ij}$	(3.1)	$A_1 = R_\perp,\quad A_2 = R_\parallel$
${}^tQ_{ijk}$	(3.3)	—
${}^rQ_{ijk}$	(3.5)	$B_2 = Q$
${}^tK^{(2)}_{ijk}$	(3.3)	—
${}^tC^{(21)}_{ijk}$	(3.4)	$B_1 = C^{(1)}_1,\quad B_2 = C^{(1)}_2$ $B_3 = C^{(1)}_3,\quad B_4 = C^{(1)}_4$
${}^tR^{(2)}_{ijk}$	(3.3)	—
${}^rK^{(2)}_{ijk}$	(3.5)	$B_2 = K^{(2)}$
${}^rC^{(21)}_{ijk}$	(3.3)	—
${}^rR^{(2)}_{ijk}$	(3.5)	$B_2 = R^{(2)}$
D_{iljk}	(3.8)	$A_1 = D_1,\quad A_2 = D_2,\quad A_3 = D_3$
${}^tC^{(22)}_{ijkl}$	(3.7)	$A_1 = C^{(2)}_1,\quad A_2 = C^{(2)}_2$ $A_3 = C^{(2)}_3,\quad A_4 = C^{(2)}_4,\quad A_5 = C^{(2)}_5$
${}^tC^{(23)}_{ijkl}$	(3.6)	—
${}^rC^{(22)}_{ijkl}$	(3.6)	—
${}^rC^{(23)}_{ijkl}$	(3.7)	$A_1 = C^{(3)}_1,\quad A_2 = C^{(3)}_2,\quad A_3 = C^{(3)}_3,$ $A_4 = C^{(3)}_4,\quad A_5 = C^{(3)}_5$
${}^tQ^{(2)}_{ijklm}$	(3.9)	—
${}^rQ^{(2)}_{ijklm}$	(3.10)	$B_1 = Q^{(2)}_1,\quad B_2 = Q^{(2)}_2,\quad B_3 = Q^{(2)}_3$

TABLE 1

In particular, if we also want symmetric irreducibility with respect to the indices i and l then

$$\begin{aligned} A_{iljk} = {} & A_1[\delta_{ij}\delta_{lk}+\delta_{ik}\delta_{lj}-2\delta_{jk}e_i e_l-2\delta_{il}e_j e_k+6e_i e_j e_k e_l] \\ & +A_2[\delta_{il}\delta_{jk}-3\delta_{jk}e_i e_l-3\delta_{il}e_j e_k+9e_i e_j e_k e_l] \\ & +A_3[\delta_{lk}e_i e_j+\delta_{jl}e_i e_k+\delta_{ik}e_l e_j+\delta_{ij}e_l e_k-4e_i e_j e_k e_l]. \end{aligned} \tag{3.8}$$

This tensor automatically satisfies $A_{iljk} = A_{jkil}$.

On the other hand we have

$$A_{ijklm} = 0, \tag{3.9}$$

$$\begin{aligned} B_{ijklm} = {} & \epsilon_{i\mu\nu}e_\nu e_\eta\{2B_1[\Delta^{(2)}_{jk,\mu\eta}(\delta_{lm}-3e_l e_m)+\Delta^{(2)}_{lm,\mu\eta}(\delta_{jk}-3e_j e_k)] \\ & +2B_2[\Delta^{(2)}_{jl,\mu\eta}\delta_{km}+\Delta^{(2)}_{kl,\mu\eta}\delta_{jm}-4\Delta^{(2)}_{lm,\mu\eta}e_j e_k \\ & +\Delta^{(2)}_{jm,\mu\eta}\delta_{kl}+\Delta^{(2)}_{km,\mu\eta}\delta_{jl}-4\Delta^{(2)}_{jk,\mu\eta}e_l e_m]\} \\ & +2B_3 e_\nu e_\eta\{\Delta^{(2)}_{jk,\mu\eta}[2\epsilon_{i\mu\nu}e_l e_m-\delta_{im}\epsilon_{l\mu\nu}-\delta_{il}\epsilon_{m\mu\nu}] \\ & +\Delta^{(2)}_{lm,\mu\eta}[2\epsilon_{i\mu\nu}e_k e_j-\delta_{ik}\epsilon_{j\mu\nu}-\delta_{ij}\epsilon_{k\mu\nu}]\}, \end{aligned} \tag{3.10}$$

provided that this tensor is symmetric and irreducible in the indices j, k and l, m and has the property $B_{ijklm} = B_{ilmjk}$.

With respect to the centre of symmetry we thus obtain the material tensors in the form given in table 1. This tells us in conjunction with (2.23) that the centre of symmetry of a transversely isotropic particle will move with the local fluid velocity if the particle is freely suspended. This, however, was to be expected.

In passing we note that the number of independent coefficients appearing in the material tensors for a particle in a second-order fluid reduces in general from 27 to 6 in going from a transversely isotropic particle to a spherically isotropic one. For a sphere of radius a another one vanishes ($C_1^{(1)}$) and the remaining five are (Brunn 1976*b*)

$$K_{\parallel} = K_{\perp} = 6\pi a, \quad R_{\parallel} = R_{\perp} = 8\pi a^3, \tag{3.11a, b}$$

$$D_1 = -\tfrac{3}{2}D_2 = \tfrac{10}{3}\pi a^3, \tag{3.11c}$$

$$C_1^{(2)} = -\tfrac{3}{2}C_2^{(2)} = \tfrac{3}{2}\pi a\,\kappa_0^{(11)}, \quad C_1^{(3)} = -\tfrac{3}{2}C_2^{(3)} = -2\pi a^3\kappa_0^{(11)}. \tag{3.11d, e}$$

Thus a spherically isotropic particle freely suspended in the fluid simply moves with the fluid, i.e. $\mathbf{u}_0 = \mathbf{U}_0$ and $\boldsymbol{\omega} = \boldsymbol{\Omega}$.

4. The behaviour of a transversely isotropic particle in a gravitational field (quiescent fluid) and in a simple shear flow

In order to draw some conclusions about the behaviour of transversely isotropic particles submerged in a viscoelastic fluid we shall consider two examples: translation through a quiescent ambient fluid under the action of external forces (gravity) and the rotation of a neutrally buoyant particle in a simple shear flow.

4.1. *The terminal state of a sedimenting particle*

Let

$$\mathbf{F}^e = F\hat{\mathbf{g}}, \quad \hat{\mathbf{g}}.\hat{\mathbf{g}} = 1, \tag{4.1}$$

denote a given external force. If the quiescent fluid is Newtonian, a transversely isotropic particle will then translate steadily without rotating, i.e.

$$\mathbf{u}_0^{(1)} = (F/\eta K_{\parallel} K_{\perp})\{K_{\parallel}\hat{\mathbf{g}} + (K_{\perp} - K_{\parallel})\hat{\mathbf{g}}.\mathbf{ee}\}, \tag{4.2a}$$

$$\boldsymbol{\omega}^{(1)} = 0. \tag{4.2b}$$

Consequently, the modifications to the state of motion of the particle in a viscoelastic fluid are governed by the equations

$$0 = -\eta^t\mathbf{K}.\mathbf{u}_0^{(2)}, \tag{4.3a}$$

$$0 = -\eta^r\mathbf{R}.\boldsymbol{\omega}^{(2)} + 2\eta K^{(2)}(\mathbf{u}_0^{(1)} \times \mathbf{e})(\mathbf{u}_0^{(1)}.\mathbf{e}) + \kappa_0^{(2)}F\hat{\mathbf{g}} \times \mathbf{u}_0^{(1)}. \tag{4.3b}$$

The solution is

$$\mathbf{u}_2^{(0)} = 0, \tag{4.4a}$$

$$\boldsymbol{\omega}^{(2)} = -A\hat{\mathbf{g}} \times (\mathbf{e} \times (\hat{\mathbf{g}} \times \mathbf{e})), \quad A = 2\left(\frac{F}{\eta}\right)^2 \frac{[K^{(2)} + \frac{1}{2}\kappa_0^{(2)}(K_{\perp} - K_{\parallel})]}{K_{\parallel}K_{\perp}R_{\perp}}. \tag{4.4b}$$

Owing to the rotation the orientation $\mathbf{e}$ of the particle will change and the rate of change as seen by an observer fixed in space is

$$\dot{\mathbf{e}} = \boldsymbol{\omega} \times \mathbf{e}. \tag{4.5}$$

If $\mathbf{e}(0)$ is the initial orientation it becomes apparent from (4.5) and (4.4) that $\mathbf{e}$ will always stay in the plane containing $\mathbf{e}(0)$ and $\hat{\mathbf{g}}$. With θ the angle between $\mathbf{e}$ and $\hat{\mathbf{g}}$, i.e. $\cos\theta = \mathbf{e}.\hat{\mathbf{g}}$, integration of (4.5) yields

$$\tan\theta = \tan\theta_0 e^{At}, \quad \cos\theta_0 = \mathbf{e}(0).\hat{\mathbf{g}}. \tag{4.6}$$

Consequently the orientation changes until it becomes either parallel ($A < 0$) or perpendicular ($A > 0$) to the direction of the external force. In either of the final states the particle translates in the direction of the force without any rotation. Since $K_{\parallel}$, $K_{\perp}$ and $R_{\perp}$ are known to be positive it is the sign of the coefficient $K^{(2)}+\frac{1}{2}\kappa_0^{(2)}(K_{\perp}-K_{\parallel})$ which determines the ultimate terminal state. For long slender bodies this coefficient is negative since a recent calculation by Leal (1975) predicted the parallel state to be the stable one.

4.2. *Rotation in simple shear*

The state of motion of a homogeneous transversely isotropic particle freely suspended in a viscoelastic fluid is characterized by the equations

$$\mathbf{u}_0 = \mathbf{U}_0, \quad \boldsymbol{\omega} = \boldsymbol{\omega}^{(1)}+\boldsymbol{\omega}^{(2)}, \tag{4.7}$$

with $$\boldsymbol{\omega}^{(1)} = \boldsymbol{\Omega}+(2Q/R_{\perp})\,\mathbf{e}\,.\,\mathbf{E}\times\mathbf{e} \tag{4.8a}$$

and $$\boldsymbol{\omega}^{(2)} = ({}^r\mathbf{R})^{-1}.\{{}^r\mathbf{R}^{(2)}\!:\tilde{\boldsymbol{\Omega}}\tilde{\boldsymbol{\Omega}}+{}^r\mathbf{C}^{(23)}\vdots\,\tilde{\boldsymbol{\Omega}}\mathbf{E}+{}^r\mathbf{Q}^{(2)}\vdots\,\mathbf{E}\mathbf{E}+(2\kappa_0^{(2)}/\eta)\,\boldsymbol{\epsilon}:(\mathbf{E}\,.\,\mathbf{S}_0^{(1)})\}. \tag{4.8b}$$

Thus, although $\boldsymbol{\omega}^{(2)}.\mathbf{e} = 0$ (i.e. as far as the rotation around the symmetry axis is concerned the particle simply rotates with the fluid) the other components of $\boldsymbol{\omega}^{(2)}$ do not vanish. This implies, by (4.5), that the orientation changes in a manner which differs from the corresponding change in a Newtonian fluid. Explicitly we have

$$\dot{\mathbf{e}} = \boldsymbol{\Omega}\times\mathbf{e}-(2Q/R_{\perp})\,[(\boldsymbol{\delta}-\mathbf{ee})\,\mathbf{e}]:\mathbf{E}-(\boldsymbol{\delta}-\mathbf{ee})\cdot\mathbf{E}\cdot[(H_2\,\boldsymbol{\delta}+H_1\,\mathbf{ee})\,\mathbf{e}]:\mathbf{E}, \tag{4.9}$$

where we have introduced the abbreviations

$$H_1 = \frac{4}{R_{\perp}}[-3Q_1^{(2)}-4Q_2^{(2)}+2Q_3^{(2)}]+\frac{2Q}{R_{\perp}}\left(\frac{2C_1^{(3)}+3C_2^{(3)}}{R_{\perp}}\right) - 2\kappa_0^{(2)}\left[\frac{1}{R_{\perp}}(6D_1+9D_2-4D_3)+\left(\frac{2Q}{R_{\perp}}\right)^2\right], \tag{4.10a}$$

$$H_2 = \left(\frac{4Q}{R_{\perp}}\right)\frac{C_1^{(3)}}{R_{\perp}}+8\frac{Q_2^{(2)}+Q_3^{(2)}}{R_{\perp}}-\kappa_0^{(2)}\left[\frac{4D_3}{R_{\perp}}-\left(\frac{2Q}{R_{\perp}}\right)^2\right]. \tag{4.10b}$$

So far, no assumptions have been made about the flow field. In order to solve (4.9), at least approximately, let us consider a simple shear flow. Choosing a space-fixed Cartesian co-ordinate system such that $\mathbf{e}_x$ denotes a unit vector in the gradient direction and $\mathbf{e}_y$ a unit vector in the flow direction, we have

$$\mathbf{E} = \tfrac{1}{2}q(\mathbf{e}_x\,\mathbf{e}_y+\mathbf{e}_y\,\mathbf{e}_x), \quad \boldsymbol{\Omega} = \tfrac{1}{2}q\mathbf{e}_z, \tag{4.11}$$

where q is the shear rate. With θ the angle between $\mathbf{e}$ and the vorticity axis $\mathbf{e}_z$ and ϕ the angle between $\mathbf{e}$ and the direction of the shear, i.e.

$$\mathbf{e} = \mathbf{e}_x\sin\theta\cos\phi+\mathbf{e}_y\sin\theta\sin\phi+\mathbf{e}_z\cos\theta, \tag{4.12}$$

the orbit equations

$$\dot{\phi} = \tfrac{1}{2}q[1+(2Q/R_{\perp})\,(1-2\cos^2\!\phi)]+\tfrac{1}{2}q^2H_1\sin^2\theta\sin\phi\cos\phi(1-2\cos^2\!\phi), \tag{4.13a}$$

$$\dot{\theta} = -(2Q/R_{\perp})\,q\sin\theta\cos\theta\sin\phi\cos\phi-q^2[\tfrac{1}{4}H_2+H_1\sin^2\!\theta\sin^2\!\phi\cos^2\!\phi]\sin\theta\cos\theta \tag{4.13b}$$

follow directly from (4.9). From (4.13*b*) we conclude that a particle with its symmetry axis in the plane of the shear will not leave that plane while a particle with its axis parallel to the vorticity axis will always maintain that orientation. Furthermore, for a spherically isotropic particle, for which $Q = H_1 = H_2 = 0$, we get $\dot{\phi} = \frac{1}{2}q$ in accordance with the experimental results of Gauthier *et al.* (1971) for a spherical particle.

Before trying to integrate these equations let us recall that even for a Newtonian fluid the behaviour of the orientation of the particle depends critically upon the sign of the coefficient $|2Q/R_\perp|-1$. Since for the important case of a spheroidal particle this quantity is negative and related to the particle axis ratio s via

$$s = \left(\frac{1-2Q/R_\perp}{1+2Q/R_\perp}\right)^{\frac{1}{2}}, \quad |2Q/R_\perp| < 1, \tag{4.14}$$

we shall confine our attention to that situation. For an arbitrary transversely isotropic particle (4.14) is interpreted as the definition of an equivalent axis ratio in terms of which the Newtonian solution to the orbit equations reads

$$\tan\phi = s\tan(2\pi t/T), \tag{4.15a}$$

$$\tan\theta = sC^{(1)}/(\sin^2\phi + s^2\cos^2\phi)^{\frac{1}{2}}, \tag{4.15b}$$

with

$$T = (2/q)\,\pi(s+s^{-1}). \tag{4.15c}$$

Thus the symmetry axis of the particle rotates about the vorticity axis,† the period of rotation being T (Jeffery orbits). The initial orientation (at $t = 0$) uniquely specifies the orbit constant $C^{(1)}$, but has no effect on T.

On the basis of the Newtonian result let us introduce new variables (τ, C) defined by the relations

$$\tan\phi = s\tan\tau, \tag{4.16a}$$

$$\tan\theta = sC/(\sin^2\phi + s^2\cos^2\phi)^{\frac{1}{2}}. \tag{4.16b}$$

If the definition (4.14) is used and the transformation (4.16) is applied to the orbit equations the variables τ and C satisfy

$$\dot{\tau} = \frac{2\pi}{T} + H_1\frac{q^2}{2}C^2\frac{(s^2\sin^2\tau-\cos^2\tau)\cos\tau\sin\tau}{1+C^2(s^2\sin^2\tau+\cos^2\tau)}, \tag{4.17a}$$

$$\dot{C} = -\tfrac{1}{4}q^2CH_2 - \frac{1+s^2}{2}q^2H_1C^3\frac{\sin^2\tau\cos^2\tau}{1+C^2(s^2\sin^2\tau+\cos^2\tau)}. \tag{4.17b}$$

It is interesting to see that C is no longer constant. In order to demonstrate that C does not merely fluctuate about its initial value but rather shows a systematic drift, let us start from an arbitrary initial state, say

$$\tau = 0, \quad C = C_0 \quad \text{at} \quad t = 0, \tag{4.18}$$

and look at the quantities $\tau^{(2)}(t)$ and $C^{(2)}(t)$ defined by

$$\tau(t) = (2\pi/T)\,t + \tau^{(2)}(t), \quad C(t) = C_0 + C^{(2)}(t). \tag{4.19}$$

† If $|2Q/R_\perp| > 1$, (4.14) furnishes an imaginary s, say is^*. Replacing s in (4.15) by is^* we get

$$\tan\phi = s^*\tanh[s^*qt/(s^{*2}-1)],$$

$$\tan\theta = s^*C^{(1)}/(s^{*2}\cos^2\phi - \sin^2\phi).$$

This is no longer periodic and the orientation ultimately attained is

$$\phi = \pm\tan^{-1}s^*, \quad \theta = \tfrac{1}{2}\pi \quad \text{if} \quad s^* \gtrless 1, \quad \text{i.e. } \pm 2Q/R_\perp < -1.$$

Clearly $(\tau^{(2)}, C^{(2)})$ will vanish for fixed t if $|qH_i| \to 0$, $i = 1, 2$, or for fixed $|qH_i|$, $i = 1, 2$, if $t \to 0$. Consequently we can always find a time $t^* > 0$ such that for all times before t^* the orientational state of the particle deviates only slightly from the Newtonian state at the same time. This allows us to approximate the quantities (τ, C) on the right-hand side of (4.17) by $((2\pi/T)t, C_0)$. The approximate forms of these equations may be readily integrated, with the result

$$\tau^{(2)}(t) = -\frac{1}{8\pi}\frac{H_1 q^2 T}{1-s^2}\left\{(1+s^2)\sin^2\left(\frac{2\pi}{T}t\right) + \frac{1+s^2+2(sC_0)^2}{(1-s^2)C_0^2}\ln\left(\frac{1+C_0^2[\cos^2((2\pi/T)t)+s^2\sin^2((2\pi/T)t)]}{1+C_0^2}\right)\right\}, \quad (4.20a)$$

$$C^{(2)}(t) = -\tfrac{1}{4}q^2C_0t\left\{H_2 + H_1\frac{(1+s^2)[2+(1+s^2)C_0^2]}{[(1-s^2)C_0]^2}\right\} + \frac{1}{4\pi}\left(\frac{1+s^2}{1-s^2}\right)H_1q^2TC_0\left\{\tfrac{1}{4}\sin\left(\frac{4\pi}{T}t\right) + \frac{[(1+C_0^2)(1+(sC_0)^2)]^{\frac{1}{2}}}{(1-s^2)C_0^2}\tan^{-1}\left[\left(\frac{1+(sC_0)^2}{1+C_0^2}\right)^{\frac{1}{2}}\tan\left(\frac{2\pi}{T}t\right)\right]\right\}. \quad (4.20b)$$

This is valid for $0 < t < t^*$, where t^* depends strongly on what we mean by a small deviation from the Newtonian state. Since the condition for a rheologically slow flow is $|qH_i| \ll 1$, $i = 1, 2$, we imagine this quantity to be so small that even after a complete rotation the orientation of the particle will be quite close to the initial orientation. Thus t^* will be of the order of T.

Since $\tau^{(2)}(T) = 0$ we conclude that if the period of rotation $\hat{T}$ is determined by measuring the time needed for the particle to perform a complete rotation around the vorticity axis then

$$\hat{T} = T. \quad (4.21)$$

For spheroids with very large or very small axis ratio s in a Newtonian fluid, where the calculated T exceeds the experimentally determined period T_e, one replaces in (4.15) the true axis ratio by an apparent axis ratio s_a which is calculated from the period measured, i.e.

$$T_e = (2\pi/q)(s_a + s_a^{-1}).$$

Since $\hat{T} = T$ the apparent axis ratio remains unchanged from its value calculated for a Newtonian fluid. In other words a graph of s_a as a function of the true axis ratio s should show no non-Newtonian influence. This agrees precisely with the experimental results of Gauthier *et al.* (1971).

As a further consequence of (4.21) we give a result following from (4.20*b*):

$$\frac{C(T)-C_0}{C_0} = \tfrac{1}{4}q^2T\left\{-H_2 + \frac{1+s^2}{(1-s^2)^2}\frac{H_1}{C_0^2}[2[(1+C_0^2)(1+(sC_0)^2)]^{\frac{1}{2}} - 2 - C_0^2(1+s^2)]\right\}. \quad (4.22)$$

Since H_1 and H_2 are independent of the orientation of the particle the expression in the curly brackets will in general be non-zero. This implies that the orbit will drift systematically until a preferred equilibrium orbit is reached ($C(T) = C_0$). From the results listed after (4.13*b*) we see that the orbits $C = 0$ and $C = \infty$ are two equilibrium

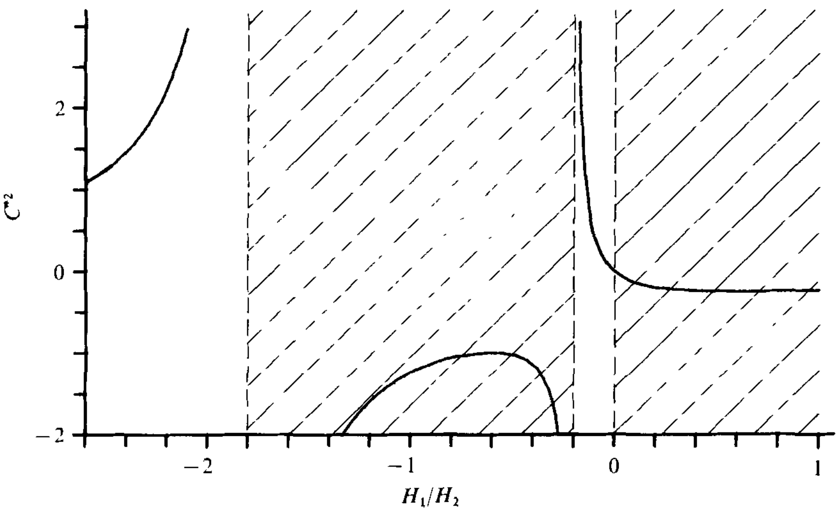

FIGURE 1. The orbit $C = C^*$ for $s = 2$. The shaded regions are physically inadmissible.

orientations for every transversely isotropic particle. Equation (4.22) furnishes a third preferred orientation C^*,

$$C^* = \begin{cases} 0 \quad \text{if} \quad H_2 = 0, \\ 2\left\{-(\ \ +s^2)\left[2+\dfrac{H_1}{H_2}+\left(\dfrac{1-s^2}{1+s^2}\right)^2\dfrac{H_2}{H_1}\right]\right\}^{-\frac{1}{2}} \quad \text{if} \quad H_2 \neq 0, \end{cases} \tag{4.23a}$$

provided that for $H_2 \neq 0$

$$-\frac{(1-s)^2}{1+s^2} < \frac{H_1}{H_2} \leqslant 0 \quad \text{or} \quad \frac{H_1}{H_2} < -\frac{(1+s)^2}{1+s^2}. \tag{4.23b}$$

If H_1/H_2 does not lie in that range C^* becomes imaginary and has to be discarded. This can also be seen from figure 1, which has been drawn for $s = 2$. Putting $C = C^*(1+\alpha)$ with $|\alpha| \ll 1$ in (4.22) reveals that C^* is stable if $H_2 \leqslant 0$. By the same token the orbits $C = 0$ and $C = \infty$ will be stable against small perturbations if

$$H_2 \geqslant 0, \quad H_2 + \frac{1+s^2}{(1+s)^2}H_1 \leqslant 0,$$

respectively. Consequently, the orbits $C = 0$ and $C = C^*$ either coincide ($H_2 = 0$) or cannot both be stable. This implies that any transversely isotropic particle can have at most two stable equilibrium orientations. The experimental results of Gauthier *et al.* (1971) indicate that long rods as well as thin disks have only one stable equilibrium orientation ($C = 0$ and $C = \infty$, respectively). There is no *a priori* reason to assume that this will generally be the case, so that one has to look at specific examples. To this end we shall consider a very simple kind of transversely isotropic particle, namely a rigid tridumbbell with two equal axes. In this way we also get an answer to the question of which will be the direction of the orbit drift.

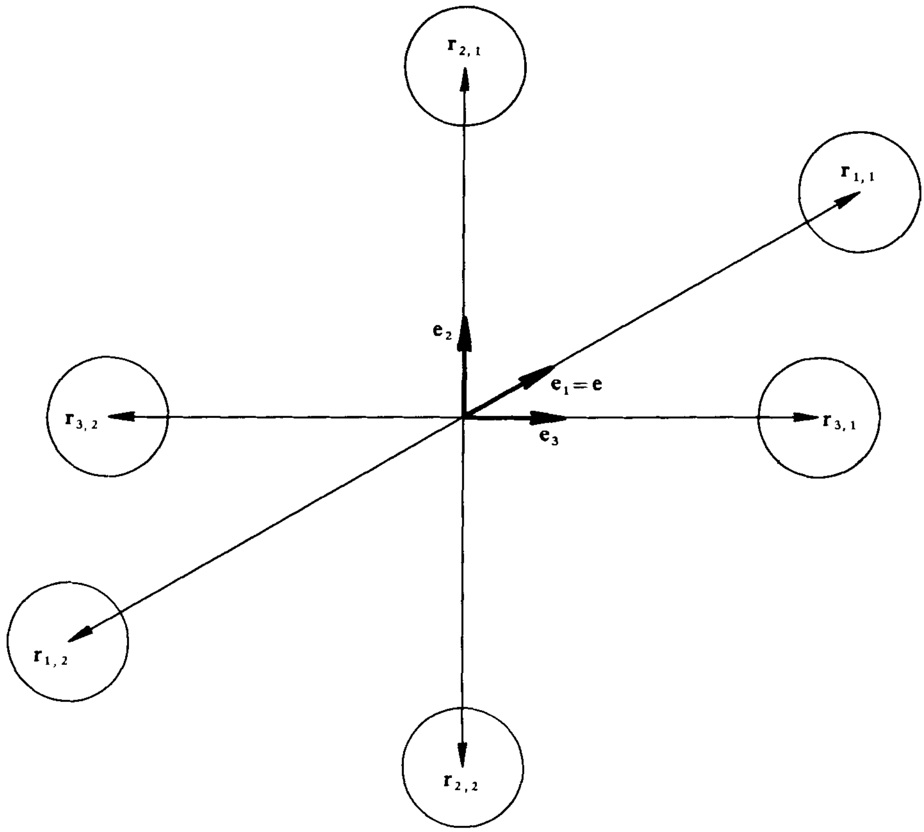

FIGURE 2. An axisymmetric tridumbbell.

5. Rigid tridumbbell

Two equal spheres of radius a joined by a thin rigid rod of negligible hydrodynamic resistance and with a centre-to-centre spacing between them of $2l$ constitute a dumbbell of length $2l$. If $\mathbf{e}$ denotes a unit vector parallel to the axis of the dumbbell the sphere centres have the co-ordinates

$$\mathbf{r}_{1/2} = \pm l\mathbf{e} \tag{5.1}$$

relative to the centre of symmetry O. A rigid tridumbbell consists of three dumbbells connected at their individual centre's of symmetry such that the co-ordinates of the sphere centres are

$$\mathbf{r}_{i,1/2} = \pm l_i \mathbf{e}_i, \quad \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \quad i,j = 1,2,3, \tag{5.2}$$

where the subscript i refers to the ith dumbbell. We shall consider only an axisymmetric tridumbbell characterized by the fact that two dumbbells, say 2 and 3, are of equal length, i.e.

$$l_2 = l_3. \tag{5.3}$$

No confusion can therefore arise if $\mathbf{e}_1$ is called $\mathbf{e}$ (see figure 2).

As long as we can neglect any hydrodynamic interaction between individual spheres we must also neglect their contribution to the torque, i.e.

$$\mathbf{G}_0 = \sum_{i=1}^{3} [\mathbf{r}_{i,1} \times \mathbf{F}_{i,1} + \mathbf{r}_{i,2} \times \mathbf{F}_{i,2}]. \tag{5.4}$$

Employing (3.11) (without $R_\parallel$, $R_\perp$, D_1 and D_2) in the transformation laws listed in appendix A and using the identity

$$\boldsymbol{\delta} - \mathbf{ee} = \mathbf{e}_2\,\mathbf{e}_2 + \mathbf{e}_3\,\mathbf{e}_3 \tag{5.5}$$

yields the desired material tensors. The non-vanishing coefficients are

$$\left.\begin{aligned}
&K_\perp = K_\parallel = 36\pi a, \quad R_\perp = 12\pi a(l_1^2 + l_2^2), \quad R_\parallel = 24\pi a l_2^2,\\
&D_1 = 6\pi a l_2^2, \quad D_2 = \tfrac{4}{3}\pi a(l_1^2 - 4l_2^2), \quad D_3 = 3\pi a(l_1^2 - l_2^2),\\
&Q = -6\pi a(l_1^2 - l_2^2), \quad C_1^{(2)} = -\tfrac{3}{2}C_2^{(2)} = 9\pi a\kappa_0^{(11)},\\
&C_1^{(3)} = -3\pi a l_1^2\kappa_0^{(11)}, \quad C_2^{(3)} = 2\pi a\kappa_0^{(11)}(2l_1^2 - l_2^2), \quad C_3^{(3)} = 3\pi a\kappa_0^{(11)}(l_1^2 - l_2^2),\\
&C_4^{(3)} = 4\pi a\kappa_0^{(11)}(l_2^2 - l_1^2), \quad C_5^{(3)} = 3\pi a\kappa_0^{(11)}(l_1^2 - l_2^2),\\
&Q_1^{(2)} = -\tfrac{4}{3}Q_2^{(2)} = \pi a\kappa_0^{(11)}(l_1^2 - l_2^2).
\end{aligned}\right\} \tag{5.6}$$

If all terms containing only the length parameter a–the terms $K_\parallel$, $K_\perp$, $C_1^{(2)}$ and $C_2^{(2)}$–are divided by 3 and l_2 is set zero the results for a dumbbell are obtained while division by 2 and putting $l_1 = 0$ furnishes the results for a bidumbbell. Since hydrodynamic interaction among the spheres has been neglected it is not surprising to find the translational behaviour of the tridumbbell to be that of an isotropic particle. This will still be true even if the restriction (5.3) that two axes be equal is dropped. As a consequence such a sedimenting tridumbbell could show no tendency to rotate but would settle by maintaining its initial orientation.

On the other hand the rotational behaviour of a rigid tridumbbell is always that of a transversely isotropic particle. In a two-dimensional shear flow of a Newtonian fluid the axis of revolution rotates periodically. The equivalent axis ratio, which appears in the equations for a Jeffery orbit and which is defined by (4.14), coincides with the true axis ratio, i.e.

$$s = l_1/l_2. \tag{5.7}$$

For a non-Newtonian fluid a drift towards the preferred orientations must occur. Since

$$H_1 = (s^2-1)\,H_2 = -2\left(\frac{s^2-1}{s^2+1}\right)^2(\kappa_0^{(2)} + \tfrac{1}{4}\kappa_0^{(11)}), \tag{5.8}$$

the restriction (4.23*b*) is violated, so that only the two equilibrium orbits $C = 0$ and $C = \infty$ exist.

On experimental grounds one expects the relation†

$$\kappa_0^{(2)} = -\tfrac{1}{2}\kappa_0^{(11)}(1+\epsilon_1), \tag{5.9}$$

with $\qquad \kappa_0^{(11)} > 0, \quad 0 \leqslant \epsilon_1 < \tfrac{2}{3},$

so that H_1 is always positive for $s \neq 1$ while H_2 is positive for $s > 1$ and negative for $s < 1$. Looking at (4.22) it is thus readily checked that for $0 < C_0 < \infty$

$$C(T) \gtrless C_0 \quad \text{if} \quad s \lessgtr 1. \tag{5.10}$$

This implies that a rigid tridumbbell whose axis ratio s is larger than one drifts towards the orbit $C = 0$ (axis of symmetry parallel to the vorticity axis) while for $s < 1$ the

† This is the mathematical way of expressing the experimental fact that the second normal-stress difference is non-positive in contrast to the positive first normal-stress difference and that the latter is much larger in magnitude than the former.

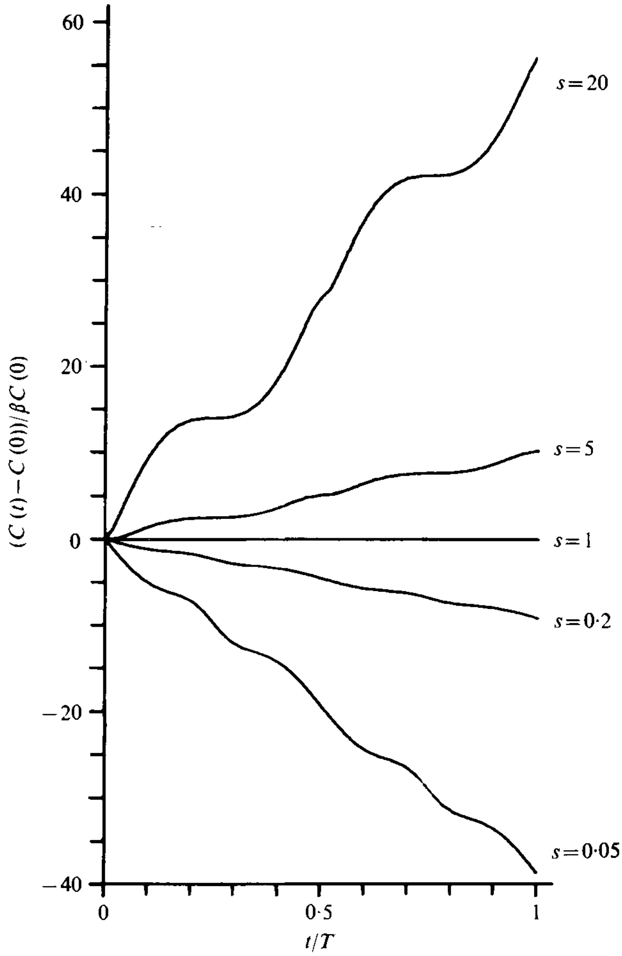

FIGURE 3. The drift of the orbit constant during one rotation of a rigid tridumbbell of axis ratio s. $C(0) = 2$, $-1 \ll \beta = q\,(\kappa_0^{(2)} + \frac{1}{4}\kappa_0^{(11)}) < 0$.

drift will be towards the orbit $C = \infty$ (axis of symmetry in the plane of the shear). The more anisotropic the tridumbbell is, i.e. the more the quantity $\frac{1}{2}(s+s^{-1})$ deviates from one, the more pronounced the drift will be. This can also be seen from figure 3, which shows the curve $C = C(t)$ as calculated from (4.20*b*). Consequently it is no surprise to find that experiments are usually carried out for particles with $s \gg 1$ and $s \ll 1$, respectively. For long cylindrical rods ($s \gg 1$) and thin disks ($s \ll 1$) the experimental results obtained at small shear rates by Gauthier *et al.* (1971) agree qualitatively with our theoretical result. That no quantitative agreement is possible may be demonstrated as follows.

If we let $s \to \infty$ the rigid tridumbbell becomes a dumbbell with $a/l_1 \approx 0$. The orbit equations (4.13) then reduce to

$$\dot{\phi} = q\cos^2\phi - \beta q \sin^2\theta \sin\phi\cos\phi(1-2\cos^2\phi), \tag{5.11a}$$

$$\dot{\theta} = q\sin\theta\cos\theta\sin\phi\cos\phi + 2\beta q\sin^3\theta\cos\theta\sin^2\phi\cos^2\phi, \tag{5.11b}$$

with

$$\beta = q(\kappa_0^{(2)} + \tfrac{1}{4}\kappa_0^{(11)}). \tag{5.12}$$

These equations are precisely the equations Leal (1975) obtained for slender, rod-like particles provided that β is replaced by $Mq(\kappa_0^{(2)}+\frac{1}{2}\kappa_0^{(11)})$.† Since M is a pure number (depending only upon the shape of the body surface) all particles which can be treated by the methods of slender-body theory (e.g. long cylindrical rods) give rise to the combination $\kappa_0^{(2)}+\frac{1}{2}\kappa_0^{(11)}$, while for the dumbbell the combination $\kappa_0^{(2)}+\frac{1}{4}\kappa_0^{(11)}$ results. Consequently no quantitative agreement between the theoretical predictions for a particle shaped like a dumbbell (or a tridumbbell) and the experimental results is possible. This contrasts with the behaviour in a Newtonian fluid, where the results depend only upon the axis ratio s.

Appendix A. Transformation laws for the material tensors

Except for ${}^t\mathbf{K}$ and ${}^t\mathbf{K}^{(2)}$, the material tensors depend upon the choice of the reference point O. In order to see this dependence let us take any other point P inside the body. If $\mathbf{r}$ denotes the directed line segment from O to P we thus have

$$\mathbf{U}_P = \mathbf{U}_0+\boldsymbol{\Omega}\times\mathbf{r}+\mathbf{r}\,.\,\mathbf{E}. \tag{A 1}$$

Since the force must be independent of the choice of the origin O while the torque and stresslet vary according to

$$\mathbf{G}_P = \mathbf{G}_0-\mathbf{r}\times\mathbf{F}, \tag{A 2a}$$

$$\mathbf{S}_P = \mathbf{S}_0-[\tfrac{1}{2}(\mathbf{rF}+\mathbf{Fr})-\tfrac{1}{3}\mathbf{r}\,.\,\mathbf{F}\boldsymbol{\delta}], \tag{A 2b}$$

the desired transformation laws follow. Using all symmetries listed in §2 and introducing $\Delta\mathbf{M}\equiv\mathbf{M}_P-\mathbf{M}_O$, where $\mathbf{M}_O$ is the form of the material tensor $\mathbf{M}$ with respect to O, these laws read‡

$$\Delta^t\mathbf{R} = {}^t\mathbf{K}\times\mathbf{r}, \quad \Delta^t\mathbf{Q} = -{}^t\mathbf{K}\mathbf{r}:\boldsymbol{\Delta}^{(2)}, \tag{A 3a, b}$$

$$\Delta^t\mathbf{C}^{(21)} = 2({}^t\mathbf{K}^{(2)}\times\mathbf{r})^{\dagger}, \quad \Delta^t\mathbf{C}^{(22)} = -2({}^t\mathbf{K}^{(2)}\mathbf{r}:\boldsymbol{\Delta}^{(2)})^{\dagger 2,4}, \tag{A 3c, d}$$

$$\Delta^t\mathbf{R}^{(2)} = [\mathbf{r}\,.\,\boldsymbol{\epsilon}\underset{2}{\cdot}\,{}^t\mathbf{K}^{(2)}\times\mathbf{r}]^{\dagger 1,2}+\tfrac{1}{2}[{}^t\mathbf{C}^{(21)}\times\mathbf{r}+({}^t\mathbf{C}^{(21)}\times\mathbf{r})^{\dagger}], \tag{A 3e}$$

$$\Delta^t\mathbf{C}^{(23)} = {}^t\mathbf{C}^{(22)}\times\mathbf{r}-({}^t\mathbf{C}^{(21)}\mathbf{r}:\boldsymbol{\Delta}^{(2)})^{\dagger 2,4}-2((\boldsymbol{\Delta}^{(2)}\,.\,\mathbf{r})\underset{2}{\cdot}\,{}^t\mathbf{K}^{(2)}\times\mathbf{r})^{\dagger 1,3}, \tag{A 3f}$$

$$\Delta^t\mathbf{Q}^{(2)} = -\tfrac{1}{2}{}^t\mathbf{C}^{(22)}\mathbf{r}:\boldsymbol{\Delta}^{(2)}-\tfrac{1}{2}((\boldsymbol{\Delta}^{(2)}\,.\,\mathbf{r})\underset{4}{\cdot}\,{}^t\mathbf{C}^{(22)})^{\dagger 1,3}+((\boldsymbol{\Delta}^{(2)}\,.\,\mathbf{r})\underset{2}{\cdot}\,{}^t\mathbf{K}^{(2)}\mathbf{r}:\boldsymbol{\Delta}^{(2)})^{\dagger 1,3}, \tag{A 3g}$$

$$\Delta^r\mathbf{R} = -\mathbf{r}\times{}^t\mathbf{R}+{}^t\mathbf{R}^{\dagger}\times\mathbf{r}-\mathbf{r}\times{}^t\mathbf{K}\times\mathbf{r}, \tag{A 4a}$$

$$\Delta^r\mathbf{Q} = -\mathbf{r}\times{}^t\mathbf{Q}-({}^t\mathbf{R}^{\dagger}-\mathbf{r}\times{}^t\mathbf{K})\,\mathbf{r}:\boldsymbol{\Delta}^{(2)}, \tag{A 4b}$$

$$\Delta^r\mathbf{K}^{(2)} = -\mathbf{r}\times{}^t\mathbf{K}^{(2)}, \tag{A 4c}$$

$$\Delta^r\mathbf{C}^{(21)} = -\mathbf{r}\times{}^t\mathbf{C}^{(21)}+2[({}^r\mathbf{K}^{(2)}-\mathbf{r}\times{}^t\mathbf{K}^{(2)})\times\mathbf{r}]^{\dagger}, \tag{A 4d}$$

$$\Delta^r\mathbf{C}^{(22)} = -\mathbf{r}\times{}^t\mathbf{C}^{(22)}-2[({}^r\mathbf{K}^{(2)}-\mathbf{r}\times{}^t\mathbf{K}^{(2)})\mathbf{r}:\boldsymbol{\Delta}^{(2)}]^{\dagger 2,4}, \tag{A 4e}$$

$$\begin{aligned}\Delta^r\mathbf{R}^{(2)} = -\mathbf{r}\times{}^t\mathbf{R}^{(2)}&+[(\boldsymbol{\epsilon}\cdot\mathbf{r})\underset{2}{\cdot}\,({}^r\mathbf{K}^{(2)}-\mathbf{r}\times{}^t\mathbf{K}^{(2)})\times\mathbf{r}]^{\dagger 1,2}\\ &+\tfrac{1}{2}({}^r\mathbf{C}^{(21)}-\mathbf{r}\times{}^t\mathbf{C}^{(21)})\times\mathbf{r}+\tfrac{1}{2}[({}^r\mathbf{C}^{(21)}-\mathbf{r}\times{}^t\mathbf{C}^{(21)})\times\mathbf{r}]^{\dagger},\end{aligned} \tag{A 4f}$$

† In Leal's notation $M=-3M_1/16\ln\epsilon$, where ϵ is the ratio of a characteristic particle thickness to the particle length and $M_1=M_1(\epsilon)$. Since $M_1>0$ in the limit $\epsilon\to 0$ the quantity

$$Mq(\kappa_0^{(2)}+\tfrac{1}{2}\kappa_0^{(11)})$$

is, like β, negative.

‡ In order to simplify the notation we denote by $\mathbf{A}\underset{k}{\cdot}\mathbf{B}$ the scalar product in which the last index of $\mathbf{A}$ is contracted with the kth index of $\mathbf{B}$ ($\mathbf{A}\underset{1}{\cdot}\mathbf{B}=\mathbf{A}\,.\,\mathbf{B}$). Also, by $\mathbf{A}^{\dagger i,j}$ we mean the tensor whose Cartesian components are obtained from those of $\mathbf{A}$ by interchanging the ith and jth indices. If this interchange concerns the last two indices, we shall merely write $\mathbf{A}^{\dagger}$.

$$\Delta{}^r\mathbf{C}^{(23)} = -\mathbf{r}\times{}^t\mathbf{C}^{(23)} + ({}^r\mathbf{C}^{(22)} - \mathbf{r}\times{}^t\mathbf{C}^{(22)})\times\mathbf{r} - 2[(\mathbf{\Delta}^{(2)}.\mathbf{r})_{\dot{2}}\,({}^r\mathbf{K}^{(2)} - \mathbf{r}\times{}^t\mathbf{K}^{(2)})\times\mathbf{r}]^{\dagger 1,3}$$
$$-[({}^r\mathbf{C}^{(21)} - \mathbf{r}\times{}^t\mathbf{C}^{(21)})\,\mathbf{r}:\mathbf{\Delta}]^{\dagger 2,4}, \quad \text{(A 4g)}$$

$$\Delta{}^r\mathbf{Q}^{(2)} = -\mathbf{r}\times{}^t\mathbf{Q}^{(2)} - \tfrac{1}{2}({}^r\mathbf{C}^{(22)} - \mathbf{r}\times{}^t\mathbf{C}^{(22)})\,\mathbf{r}:\mathbf{\Delta}^{(2)} - \tfrac{1}{2}[(\mathbf{\Delta}^{(2)}.\mathbf{r})_{\dot{4}}\,({}^r\mathbf{C}^{(22)} - \mathbf{r}\times{}^t\mathbf{C}^{(22)})]^{\dagger 1,3}$$
$$+[(\mathbf{\Delta}^{(2)}\cdot\mathbf{r})_{\dot{2}}\,({}^r\mathbf{K}^{(2)} - \mathbf{r}\times{}^t\mathbf{K}^{(2)})\,\mathbf{r}:\mathbf{\Delta}^{(2)}]^{\dagger 1,3}, \quad \text{(A 4h)}$$

$$\Delta\mathbf{D} = \mathbf{\Delta}^{(2)}:(\mathbf{r}{}^t\mathbf{K}\mathbf{r}):\mathbf{\Delta}^{(2)} - \mathbf{\Delta}^{(2)}:\mathbf{r}{}^t\mathbf{Q} - ({}^t\mathbf{Q}^{\dagger 1,3}\mathbf{r}):\mathbf{\Delta}^{(2)}, \quad \text{(A 5)}$$

where all tensors on the right-hand sides are evaluated at O.

In those equations $\mathbf{\Delta}^{(2)}$ denotes the isotropic tensor of the fourth rank

$$(\mathbf{\Delta}^{(2)})_{ijkl} = \Delta^{(2)}_{ij,kl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \tfrac{1}{3}\delta_{ij}\delta_{kl}, \quad \text{(A 6)}$$

which is symmetric and irreducible in the pairs of indices i, j and k, l.

Appendix B. The $\kappa_0^{(2)}$ material tensors

Although the concept of material tensors in general will not be applicable to terms involving $\kappa_0^{(2)}$ [time derivatives appear in (2.19) and (2.20)] there are quite a few cases where this concept does work. Besides the obvious situation of a particle at rest, we have listed in § 2 two more of these exceptional cases. There no decomposition of the problem into $\kappa_0^{(11)}$ and $\kappa_0^{(2)}$ contributions is necessary, since the material tensors will split up into such terms. If we retain for the $\kappa_0^{(2)}$ tensors the same notation as in (2.14) and (2.15) respectively, the following relation between the $\kappa_0^{(2)}$ tensors and the Newtonian material tensors results:

$$^tK^{(2)}_{ijk} = 0, \quad {}^tC^{(21)}_{ijk} = -\kappa_0^{(2)}\epsilon_{ij\mu}{}^tK_{\mu k}, \quad {}^tC^{(22)}_{ijkl} = \kappa_0^{(2)}\Delta^{(2)}_{jk,i\mu}{}^tK_{\mu l}, \quad \text{(B 1a–c)}$$
$$^tR^{(2)}_{ijk} = -\tfrac{1}{2}\kappa_0^{(2)}[\epsilon_{ij\mu}{}^tR_{\mu k} + \epsilon_{ik\mu}{}^tR_{\mu j}], \quad \text{(B 1d)}$$
$$^tC^{(23)}_{ijkl} = \kappa_0^{(2)}[-\epsilon_{il\mu}{}^tQ_{\mu jk} + \Delta^{(2)}_{jk,i\mu}{}^tR_{\mu l}], \quad \text{(B 1e)}$$
$$^tQ^{(2)}_{ijklm} = \tfrac{1}{2}\kappa_0^{(2)}[\Delta^{(2)}_{jk,i\mu}{}^tQ_{\mu lm} + \Delta^{(2)}_{lm,i\mu}{}^tQ_{\mu jk}], \quad \text{(B 1f)}$$
$$^rK^{(2)}_{ijk} = -\kappa_0^{(2)}\Delta^{(2)}_{jk,\mu\nu}\epsilon_{i\mu\eta}{}^tK_{\eta\nu}, \quad \text{(B 1g)}$$
$$^rC^{(21)}_{ijk} = -\kappa_0^{(2)}[\epsilon_{ik\mu}{}^tR_{\mu j} + \epsilon_{ij\mu}{}^tR_{k\mu}], \quad \text{(B 1h)}$$
$$^rC^{(22)}_{ijkl} = -\kappa_0^{(2)}[\epsilon_{il\mu}{}^tQ_{\mu jk} + 2\Delta^{(2)}_{jk,\mu\nu}\epsilon_{i\mu\eta}{}^tQ_{l\nu\eta}], \quad \text{(B 1i)}$$
$$^rR^{(2)}_{ijk} = -\kappa_0^{(2)}\Delta^{(2)}_{jk,\mu\nu}\epsilon_{i\mu\eta}{}^rR_{\eta\nu}, \quad \text{(B 1j)}$$
$$^rC^{(23)}_{ijkl} = -\kappa_0^{(2)}[\epsilon_{il\mu}{}^rQ_{\mu jk} + 2\Delta^{(2)}_{jk,\mu\nu}\epsilon_{i\mu\eta}{}^rQ_{l\nu\eta}], \quad \text{(B 1k)}$$
$$^rQ^{(2)}_{ijklm} = \kappa_0^{(2)}\epsilon_{i\mu\nu}[\Delta^{(2)}_{jk,\nu\eta}D_{\eta\mu lm} + \Delta^{(2)}_{lm,\nu\eta}D_{\eta\mu jk}]. \quad \text{(B 1l)}$$

Thus, while there is never a $\kappa_0^{(2)}$ contribution to ${}^t\mathbf{K}^{(2)}$, there is also no $\kappa_0^{(2)}$ contribution to ${}^t\mathbf{R}^{(2)}$ and to ${}^r\mathbf{C}^{(21)}$ for all those particles for which translational and rotational motion is uncoupled in a Newtonian fluid. In particular, this is true for transversely isotropic particles, for which the tensors given by (B 1a–l) are of the same form as the ones listed in table 1 in § 3. This time the coefficients appearing there are

$$C_1^{(1)} = -\kappa_0^{(2)}K_\perp, \quad C_2^{(1)} = \kappa_0^{(2)}(K_\perp - K_\parallel), \quad C_1^{(2)} = -\tfrac{3}{2}C_2^{(2)} = \tfrac{1}{2}\kappa_0^{(2)}K_\perp, \quad \text{(B 2a–c)}$$
$$C_5^{(2)} = -\tfrac{3}{2}C_4^{(2)} = \tfrac{1}{2}\kappa_0^{(2)}(K_\parallel - K_\perp), \quad K^{(2)} = -\tfrac{1}{2}\kappa_0^{(2)}(K_\parallel - K_\perp), \quad R^{(2)} = -\tfrac{1}{2}\kappa_0^{(2)}(R_\parallel - R_\perp), \quad \text{(B 2d–f)}$$
$$C_1^{(3)} = -\tfrac{1}{2}C_2^{(3)} = -C_3^{(3)} = \tfrac{1}{2}C_4^{(3)} = -\tfrac{1}{2}C_5^{(3)} = \kappa_0^{(2)}Q, \quad \text{(B 2g)}$$
$$Q_1^{(2)} = \kappa_0^{(2)}(D_1 + \tfrac{3}{2}D_2), \quad Q_2^{(2)} = -\tfrac{1}{2}\kappa_0^{(2)}D_3. \quad \text{(B 2h, i)}$$

This can serve as an alternative check on the correctness of (4.4) and (4.9) respectively.

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